Abstract. The basic aim of all investors investing in securities is to achieve maximum yield while keeping their loss risks at a minimum. A number of techniques and methods have been devised as a decision support tool in this domain. In this paper, the initial model of portfolio optimization has been enhanced by using computer simulation. It was used in the model to generate random rates of return from the bivariate normal distribution. It was assumed that its parameters are established on the basis of empirical data and estimates of the expected rates of return for securities. In this model, the efficient solution set, from which an optimum portfolio is derived, is obtained by finding extrema of the function by means of the Lagrange method.

Keywords. Portfolio optimization, rate of return, computer simulation, bivariate normal distribution, random numbers generating

1. Introduction

Companies and individuals have a range of possibilities to invest their financial means, at which they face a variety of dilemmas. One way of doing this is to invest in securities, which has become very popular in the wider region in recent years. Through this type of investment an investor can make significant profits in a relatively short period of time, however, turbulent market conditions, influenced by a range of factors, can result in completely opposite outcomes. Obviously, the major motivation for investment in securities is maximizing the yield. Nevertheless, every sensible investor will endeavour to minimize the risks associated with securities business. It can be logically expected that these aims are more likely to be achieved by acquiring a smaller quantity of shares in different companies than by investing the whole disposable amount in securities of a single company. These assumptions are the foundation of portfolio management.

The portfolio selection theory was pioneered in 1950s by H.M. Markowitz. For his contributions in this field he was awarded the Nobel Prize for Economics in 1990. According to his theory, an investor will be prone to choose a portfolio that provides a maximum yield for a given level of risk, i.e. a minimum risk for a given level of yield. A set of all such portfolios is called an efficient set. Ultimately, the investor will choose from the efficient set the portfolio which best meets his requirements.

To increase their efficiency, portfolio optimization models have been modified and supplemented on a number of occasions over the years. We had the same objective in mind when we implemented computer simulation in the initial portfolio optimization model as presented in this paper. In the proposed model computer
simulation was used to generate random rates of return. Because of its characteristics, the bivariate normal distribution was chosen for this purpose. It was assumed that its parameters are determined on the basis of empirical data and estimates of the expected rates of return for securities. In this model, establishing the efficient solution set, from which an optimum portfolio is chosen, comes down to finding extrema of the function. For this purpose the method of Lagrange multipliers was used.

2. Basic assumptions and formulation of the model

In its simplified form, portfolio management is based on an assumption that an investor has at his disposal a fixed amount of capital, all of which he wishes to invest into buying a certain quantity of securities, and that he has aversion to risk. His objective is to maximize the total returns resulting from such diversified investment. The problem, however, is that securities with the highest expected rate of return are usually the riskiest.

If it is assumed that there are $n$ types of securities considered for acquisition by the investor, then the sum of all individual investments ($W_i$) has to correspond to the total amount of capital ($C$) intended for investing:

$$\sum_{i=1}^{n} W_i = C, \quad W_i \geq 0.$$

An estimated rate of return ($X_i$) for security $i$ is defined as

$$X_i = \frac{\text{estimated selling price} + \text{estimated dividend} - \text{purchase price}}{\text{purchase price}}.$$

The actual rate of return ($R_i$) for security $i$ at the time of selection of portfolio is not known, so is considered to be a random variable with mean $\mu(R_i) = X_i$. It follows from the above that the actual rate of return (RP) after a certain time is

$$RP = \sum_{i=1}^{n} W_i X_i.$$

The actual rate of the portfolio is also a random variable, with mean

$$Z = \sum_{i=1}^{n} W_i X_i.$$

The investor can choose between different securities combinations to form a portfolio that is best suited to his preferences. A more cautious investor will choose a portfolio with lower expected return rather than risk a loss. Investors who want higher returns will have a higher preference for risk. Thus, the expected return ($Z$) and the standard deviation of this return ($\sqrt{V}$), as a measure of risk, represent two main criteria upon which portfolio choice is based. Their relationship is shown in Figure 1.

![Figure 1. Expected returns and standard deviations of different portfolios](image)

An investor will deem as acceptable only portfolios that correspond to values in the arc AC. Portfolios outside this segment either have a higher standard deviation of return, i.e. they are riskier (the lowest risk is demonstrated by portfolio A), or they result in a lower expected return (the highest expected return is with portfolio C). Portfolio B satisfies both conditions equally well, whereas D, E and F are not acceptable according to any of the criteria.

The variance of the expected return of the analysed portfolio ($V$) can be calculated as follows:

$$V = \sum_{i=1}^{n} W_i^2 \sigma_i^2 + C^2 \sigma_e^2,$$

where

- $W_i$ = amount invested in security $i$
- $\sigma_i$ = standard deviation of the expected rate of return for security $i$
- $C$ = amount available for investment
- $\sigma_e$ = standard error
The value of variance, i.e. of standard deviation of portfolio return, needs to be minimized. To do this, the following restrictions have to be met:

\[ \sum_{i=1}^{n} W_i = C \]
\[ \sum_{i=1}^{n} W_i X_i = Z \]
\[ W_i \geq 0, \quad i = 1, 2, ..., n \]

The first two conditions are already explained, whereas the last one, also mentioned previously, represents the requirement of non-negativity for the capital invested.

Searching for a minimum variance of the expected return of a portfolio comes down to calculating extrema of the function by means of the Lagrange method:

\[
\frac{\partial V}{\partial W_1} = \frac{\partial f}{\partial W_1} - \kappa \frac{\partial g}{\partial W_1} - \lambda \frac{\partial h}{\partial W_1} = 0
\]
\[
\frac{\partial V}{\partial W_2} = \frac{\partial f}{\partial W_2} - \kappa \frac{\partial g}{\partial W_2} - \lambda \frac{\partial h}{\partial W_2} = 0
\]
\[
\vdots
\]
\[
\frac{\partial V}{\partial W_n} = \frac{\partial f}{\partial W_n} - \kappa \frac{\partial g}{\partial W_n} - \lambda \frac{\partial h}{\partial W_n} = 0
\]

In this way, together with two initial conditions, we arrive at a system of \( n + 2 \) equations with equal number of unknowns, where \( \kappa \) and \( \lambda \) represent Lagrange multipliers. The system of equations is posed and solved for all the analyzed values of expected return \( Z \). This procedure is repeated until for each variable \( W_i \) the requirement of non-negativity is met.

### 3. Generating random securities rates of return from the bivariate normal distribution

In the model it is assumed that securities rates of return are generated as random numbers from the bivariate normal distribution using a computer.

The probability density function of the bivariate normal distribution, with parameters \( \mu_1, \mu_2, \sigma_1, \sigma_2 \) and \( \rho \), is

\[
f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{(x_1-\mu_1)^2 + 2\rho(x_1-\mu_1)(x_2-\mu_2) + (x_2-\mu_2)^2}{2(1-\rho^2)}},
\]

where

\[
u = \frac{(x_1-\mu_1)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2) + (x_2-\mu_2)^2}{\sigma_1\sigma_2},
\]

\[
\rho = \text{cor}(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma_1\sigma_2},
\]

for

\[-\infty < x_1 < \infty, \quad -\infty < x_2 < \infty.\]

Parameters must satisfy following conditions:

\[\sigma_1 > 0, \quad \sigma_2 > 0 \quad \text{and} \quad -1 < \rho < 1.\]

If the bivariate continuous random variable \((X_1, X_2)\) follows a normal distribution, the marginal probabilities are

\[
f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2, \quad x_1 \in \mathbb{R},
\]

\[
f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_1, \quad x_2 \in \mathbb{R}.
\]

In this paper simulations of rates of return are based on the conditional distribution. The conditional distribution of the random variable \(X_2\), given \(X_1 = x_1\), is

\[
f_{X_2 | X_1 = x_1}(x_2) = \frac{f(x_1, x_2)}{f_1(x_1)}, \quad x_2 \in \mathbb{R}.
\]

This conditional distribution also follows a normal distribution with parameters

\[
\mu(X_2 | X_1 = x_1) = \mu_2 + \frac{\sigma_2}{\sigma_1} (x_1 - \mu_1),
\]

\[
\sigma(X_2 | X_1 = x_1) = \sigma_2 \sqrt{1 - \rho^2}.
\]

The procedure of generating random rates of return from the bivariate normal distribution is formalized by a flow diagram shown in Figure 2.
In the process of generating random rates of return it is first necessary to determine parameters of the bivariate normal random variable \((X_1, X_2)\). After that, the intended number of simulations is established, followed by generating the first random number \(p_1\) from the interval \((0,1)\). Adequate computer support is then used to determine the value \(x_1\) of random variable \(X_1\) for which \(P(X_1 \leq x_1) = p_1\). This value in the conditional distribution represents a fixed level of random variable \(X_1\). On its basis, and using the above mentioned formulas, in the next step the parameters of conditional distribution are determined. After that, another random number is generated from the interval \((0,1)\). Simulated security rate of return \(x\) of random variable \(X \sim N(\mu(X_2|X_1=x_1), \sigma(X_2|X_1=x_1))\) represents the value for which \(P(X \leq x) = p_2\). The described procedure is repeated \(n\) times for each of the securities considered for acquisition. The data obtained in this way are the basis for forming the associated frequency distributions of simulated values.

4. An example of portfolio optimization based on computer simulation of securities rates of return

Let it be assumed that an investor has the amount of 100000 EUR at his disposal that he intends to invest in shares of three companies. Let it further be assumed that on the basis of data on share price movement from previous periods the mean rates of return (\(\mu\)) and associated standard deviations (\(\sigma\)) could be calculated. In addition, the investor has estimated the expected rate of return (\(\mu_2\)) and standard deviations (\(\sigma_2\)) of the variables thus defined for the future period. These data, together with the associated correlation coefficients (\(\rho\)), are listed in Table 1.

<table>
<thead>
<tr>
<th>SHARE</th>
<th>(\mu_1)</th>
<th>(\sigma_1)</th>
<th>(\mu_2)</th>
<th>(\sigma_2)</th>
<th>(\rho)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.9</td>
<td>0.2</td>
<td>0.7</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>0.3</td>
<td>0.6</td>
<td>0.1</td>
<td>0.7</td>
</tr>
<tr>
<td>C</td>
<td>0.8</td>
<td>0.4</td>
<td>1.2</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 1. Basic parameters of rates of return distributions for three hypothetical shares

The values established for each of the shares represent the parameters of the bivariate normal distribution from which random rates of return will be generated.

For each of the shares considered for acquisition it is necessary to determine how many simulations of rates of return need to be carried out. The next step in generating the first rate of return for share A are consists in choosing a random number from the interval \((0,1)\). Using a computer random number generator the obtained number was 0.6217. To this probability of random variable \(X_1\), which follows normal distribution with parameters \(\mu_1 = 0.9\) and \(\sigma_1 = 0.2\), corresponds the value of 0.962. Consequently, the following applies:

\[ P(X_1 \leq 0.962) = 0.6217. \]
The conditional distribution parameters of random variable \( X_2 \), given \( X_1 = 0.962 \), are

\[
\mu(X_2 | X_1 = 0.962) = 0.7 + 0.8 \cdot \frac{0.2}{0.2} = 0.962 - 0.9 = 0.7496,
\]

\[
\sigma(X_2 | X_1 = 0.962) = 0.2 \sqrt{1 - 0.8^2} = 0.12.
\]

After calculating the conditional distribution parameters, it is again necessary to generate a random number from the interval \((0, 1)\). A random number generator has thus produced the value of 0.3572. To this conditional distribution probability of random variable \( X \), which follows normal distribution with parameters \( \mu(X_2 | X_1 = 0.962) = 0.7496 \) and \( \sigma(X_2 | X_1 = 0.962) = 0.12 \), corresponds the value of 0.7057, i.e.

\[
P(X \leq 0.7057) = 0.3572.
\]

Assuming that the random variable defined as rate of return for shares of Company A follows the bivariate normal distribution, its first simulated value is determined in this way. The process is repeated until other random rates of return for share A have been generated as well, after which the associated frequency distribution is formed. Frequency distributions of simulated rates of return for shares of Company B and C are formed in the same way. They were the basis for calculating the expected rate of return and associated standard deviation.

<table>
<thead>
<tr>
<th>SHARE</th>
<th>EXPECTED RATE OF RETURN</th>
<th>STANDARD DEVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.72</td>
<td>0.19</td>
</tr>
<tr>
<td>B</td>
<td>0.59</td>
<td>0.13</td>
</tr>
<tr>
<td>C</td>
<td>1.25</td>
<td>0.55</td>
</tr>
</tbody>
</table>

Table 2. Expected rates of return and standard deviations

It can be easily concluded that in the analyzed example the expected return will vary between 59000 EUR, if all the money assets are invested in the shares of Company B, and 125000 EUR, if the investor decides to buy only the shares of Company C.

The investor has estimated that the common component of error has a standard deviation of 0.1, and has decided that the expected return has to be at least 110000 EUR. Consequently, he does not want to consider any possibilities earning him a lower return than the one stated above. At the same time, since he is risk-averse, the investor wishes to determine a portfolio with minimum variance (standard deviation) for every level of return. According to the previously stated formula, this means that it is necessary to minimize

\[
V = 0.19^2 W_1^2 + 0.13^2 W_2^2 + 0.55^2 W_3^2 + 0.1^2 100000^2
\]

subject to

\[
W_1 + W_2 + W_3 = 100000
\]

\[
0.72 W_1 + 0.59 W_2 + 1.25 W_3 = 110000
\]

\[
W_1 \geq 0, W_2 \geq 0, W_3 \geq 0
\]

By applying the Lagrange method the following is obtained first:

\[
V = 0.19^2 W_1^2 + 0.13^2 W_2^2 + 0.55^2 W_3^2 + 0.1^2 100000^2
\]

\[
- \kappa (W_1 + W_2 + W_3 - 100000)
\]

\[
- \lambda (0.72 W_1 + 0.59 W_2 + 1.25 W_3 - 110000)
\]

The minimization now follows the standard method of differential calculus, and after derived function with respect to each variable the following system of equations is formed:

\[
2 \cdot 0.19^2 W_1 - \kappa - 0.72 \lambda = 0
\]

\[
2 \cdot 0.13^2 W_2 - \kappa - 0.59 \lambda = 0
\]

\[
2 \cdot 0.55^2 W_3 - \kappa - 1.25 \lambda = 0
\]

\[
W_1 + W_2 + W_3 = 100000
\]

\[
0.72 W_1 + 0.59 W_2 + 1.25 W_3 = 110000
\]

Solving this system of equations yields the following results:

\[
W_1 = 8364127, W_2 = -444392, W_3 = 607979.93, \\
\kappa = -35726.33, \lambda = 58007.26.
\]

Since \( W_2 \) has negative value (thus it does not meet the requirement of non-negativity) the procedure is repeated for \( W_2 = 0 \). In that case the function of the following form is minimized:

\[
V = 0.19^2 W_1^2 + 0.55^2 W_3^2 + 0.1^2 100000^2
\]

\[
- \kappa (W_1 + W_3 - 100000)
\]

\[
- \lambda (0.72 W_1 + 1.25 W_3 - 110000)
\]

By solving the system of equations obtained in this way we get the results that satisfy all the conditions set by the model:

\[
W_1 = 2830189, W_3 = 71698.11, \\
\kappa = -54108.4, \lambda = 77988.61.
\]

The procedure is repeated in the identical way for the values of expected return \((Z)\) of 115000 EUR, 120000 EUR and 125000 EUR (it is assumed that these amounts were arbitrarily
The values $W_1$, $W_2$, and $W_3$ that need to be invested in order to achieve the expected return are listed in Table 3.

<table>
<thead>
<tr>
<th>PORTFOLIO</th>
<th>Z</th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$W_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>110000</td>
<td>28301.89</td>
<td>0</td>
<td>71698.11</td>
</tr>
<tr>
<td>2</td>
<td>115000</td>
<td>18867.92</td>
<td>0</td>
<td>81132.08</td>
</tr>
<tr>
<td>3</td>
<td>120000</td>
<td>9433.96</td>
<td>0</td>
<td>90566.04</td>
</tr>
<tr>
<td>4</td>
<td>125000</td>
<td>0</td>
<td>0</td>
<td>100000.00</td>
</tr>
</tbody>
</table>

Table 3. Expected returns and amounts invested in particular portfolios

The key criterion for choosing an optimal portfolio is the risk assessment that goes with such investment. Therefore, when making decisions in this domain, it is necessary to determine the standard deviation of the expected return for each portfolio. According to the above formula, standard deviation of the expected return for the first portfolio amounts to

$$\sqrt{V} = \sqrt{0.19^2 \times 28301.89^2 + 0.55^2 \times 71698.11^2 + 0.1^2 \times 100000^2}$$

$$\sqrt{V} = 41035.999$$

Apart from standard deviations of the expected returns for the determined portfolios, Table 4 lists also their minimum expected returns (likely minimum). They represent the lowest return to be expected for a given portfolio at probability of 0.9733.

<table>
<thead>
<tr>
<th>PORTFOLIO</th>
<th>STANDARD DEVIATION ($\sqrt{V}$)</th>
<th>LIKELY MINIMUM ($Z-2\sqrt{V}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41035.999</td>
<td>27928.002</td>
</tr>
<tr>
<td>2</td>
<td>45869.727</td>
<td>23260.546</td>
</tr>
<tr>
<td>3</td>
<td>50836.804</td>
<td>18326.392</td>
</tr>
<tr>
<td>4</td>
<td>55901.699</td>
<td>13196.602</td>
</tr>
</tbody>
</table>

Table 4. Results of risk analyses

As the expected return for a portfolio increases, its standard deviation grows as well, since in this way the amount of risky shares in the portfolio goes up. Ultimately, the investor can base his choice on a minimum expected return. In accordance with this, in this example a risk-averse investor would decide on the first portfolio.

5. Conclusions

This paper discusses a possible approach to solving the problem of portfolio optimization. Although securities market is highly volatile, the presented model allows us to a certain degree to reduce the uncertainty associated with portfolio composition. Nevertheless, this does not mean that we can disregard possible impacts of different factors that are not included in the model.

The specific feature of the proposed model is generating random rates of return from the bivariate normal distribution. Simulating the rate of return for shares enhances the usage value of the model. In its design and implementation a crucial place is held by computer support.

6. References


